Kinetic Equations

Text of the Exercises

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Exercise 1

Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Assume that φ is **collision invariant**, i.e.

$$\varphi\left(v'\right) + \varphi\left(v'_*\right) = \varphi\left(v\right) + \varphi\left(v_*\right) \tag{1}$$

for all $v,\ v_* \in \mathbb{R}^3$, $\omega \in \mathbb{S}^2$, and with v' and v'_* defined as:

$$\begin{cases} v' = v + (v_* - v) \cdot \omega\ \omega, \\ v'_* = v_* - (v_* - v) \cdot \omega\ \omega. \end{cases} \tag{2}$$

- Assume additionally that φ vanishes on $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(-1,0,0)$. Prove that φ is zero on $\mathbb{Z}^3$.
- Under the same assumption of the previous point, prove that actually φ is zero on $\mathbb{R}^3$.

  Hint: Denote $a = (1/2, 1/2, 0)$, $b = (1/2, -1/2, 0)$, $c = (-1/2, -1/2, 0)$ and $d = (-1/2, 1/2, 0)$, what can be said about $\varphi(a) + \varphi(b)$, $\varphi(b) + \varphi(c)$, $\varphi(c) + \varphi(d)$, $\varphi(d) + \varphi(a)$? What about $\varphi(a) + \varphi(c)$? Iterate this idea and use continuity to conclude.
- Consider now a generic continous φ which is collision invariant. Use the previous point to prove that there exist $a,\ c \in \mathbb{R}$ and $b \in \mathbb{R}^3$ such that

  $$\varphi(v) = a\,|v|^2 + b \cdot v + c, \tag{3}$$

  for any $v \in \mathbb{R}^3$.

Remark. *Notice that despite the similarities with the result presented in class, the final result is here achieved under much less regularity assumptions.*

Exercise 2

Let (X, Σ, μ) be a finite measure space. Let $f : X \to X$ a **measure-preserving transformation**, i.e., a mapping such that for any $A \in \Sigma$ we have $\mu\left(f^{-1}(A)\right) = \mu(A)$. For any $x \in X$ and $A \in \Sigma$, we say that x is **recurrent with respect to** A if $\left|\left\{k \in \mathbb{N} \middle|\ f^k(x) \in A\right\}\right| = +\infty$, where $f^{k+1}(x) := f\left(f^k(x)\right)$.

Prove the **Poincaré recurrence Theorem**, i.e., prove that for any measurable set $A \in \Sigma$ almost every point of A is recurrent with respect to A.

Discuss then how this would seem to contradict the H-theorem (this is the so-called Zermelo's Paradox).

Hint: Consider the family of sets $U_p := \bigcup_{k \geqslant p} f^{-k}(A)$. Can we express the set of non-recurrent points in term of $\{U_p\}_{p \in \mathbb{N}}$?

Exercise 3

We will now study a toy model, useful to understand the Zermelo's Paradox.

Consider the following setting. We have N points on a ring, with N a large integer number. At every point there is a ball, that can be either white or black. Between every couple of points there is an edge that can contain or not contain a marker. We consider that the system evolves in discrete times according to the following rule: at each step, the balls rotate of one position (the ball in position 1 goes to position 2, 2 to 3 and so on, and finally the ball in position N goes to position 1). If the ball encounters a marker on the edge, it changes its color.

- Let $W(t)$ the total number of white balls and $B(t)$ the total number of black balls at time t. Let $w(t)$ (and $b(t)$ respectively) the number of white (and respectively black) balls that will cross a marker at the next step.

  Let $\Delta(t) := B(t) - W(t)$. Describe $\Delta(t+1)$ in terms of $\Delta(t)$.

  Let in addition μ be the fraction of markers over the total number of edges. Assume moreover that $\frac{w(t)}{W(t)} = \frac{b(t)}{B(t)} = \mu$ (this corresponds to the Stosszahlansatz). Find an explicit formula for $\Delta(t)$ in terms of $\Delta(0)$ and μ.

- Denote with $X_j(t)$ the color of the ball at position j at time t, where $X_j(t) = 1$ if the ball is black and $X_j(t) = -1$ if the ball is white. Denote with m_j the fact that a marker is or is not on the edge between position j and position $j+1$, where $m_j = 1$ if there is no marker (and the ball does not change colour) while $m_j = -1$ if there is a marker (and the ball does change colour).

  Describe $\Delta(t)$ in terms of $\{X_j(0)\}_{j=1}^N$ and $\{m_j\}_{j=1}^N$.

- Suppose now that every edge has a probability $0 \leqslant \mu \leqslant 1$ of having a marker. Denote with $\langle\cdot\rangle$ the expectation over all the possible configurations of markers, meaning that if M is the set of all the possible configurations and we denote with $m = \{m_j\}_{j=1}^N$ one such configuration, for $f : M \to \mathbb{R}$, we define

$$\langle f \rangle := \frac{1}{|M|} \sum_{m \in M} f(m). \tag{4}$$

  Given that for any $t > 0$ the value $\Delta(t)$ depends on the configuration of markers on the edges, we can calculate its expectation.

  Prove that for any $t < N$, $\langle \Delta(t) \rangle = (1 - 2\mu)^t \Delta(0)$.

- Discuss the link between the quantity we just obtained and the H-theorem for large values of N.

- Notice that the evolution of the system is reversible, that $\Delta(t)$ is periodic and find the period. Discuss how this solves the Zermelo's Paradox.